Now, define the height of a_k^j to be $j+k$. Then a_1^1 is the only element of height 2; likewise a_2^1 and a_1^2 are the only elements of height 3, and so on.

Since for any positive integer $m \geq 2$ there are only $m-1$ elements of height m,

Then, arrange the elements of $\bigcup_{n=1}^{\infty} A_n$

according to their height as $a_1^1, a_1^2, a_2^1, a_3^1, a_2^2, a_1^3, a_1^4, \dots,$ being careful to remove any a_k^j that has already been counted.

Similarly, to arrange listing the elements of $\bigcup_{n=1}^{\infty} A_n$ in the following array and counting them in the order indicated by the arrows:

$$\begin{array}{ccccc} a_1^1 & a_2^1 & a_3^1 & a_4^1 & \cdots \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & \cdots \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & \cdots \\ a_1^4 & a_2^4 & \cdots & & \end{array}$$

Hence this counting scheme eventually counts every a_k^j proves that $\bigcup_{n=1}^{\infty} A_n$ is countable.

Hence countable union of countable sets is countable.

Hence proved the theorem.

1.5 G Corollary :

The set of all rational numbers is countable.

Proof : The set of all rational numbers is the union $\bigcup_{n=1}^{\infty} E_n$ where E_n is the set of rationals which can be written with denominator n.

That is, $E_n = \{ 0/n, -1/n, 1/n, -2/n, 2/n, \ldots \}$.

Now, each E_n is clearly equivalent to the set of all integers.

Since, the set of all integers is countable.

So, E_n is countable.

~~Hence the set of all rationals is the countable union of~~

By our known Theorem "The countable union of countable sets is countable"

Hence the set of all rationals is the countable union of countable sets.

1.5.H Theorem:

If B is an infinite subset of the countable set A, then B is countable.

Proof:

Let $A = \{a_1, a_2, \dots\}$ is countable set, and B is an infinite subset of A.

Then each element of b is an a_i.

Let n_1 be the smallest index for which $a_{n_1} \in B$.

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Let n_2 be the next smallest index, and so on.

Then $B = \{a_{n_1}, a_{n_2}, \ldots\}$.

The elements of B are thus labeled with 1, 2, ... and so B is countable.

1.5 I Corollary:

The set of all rational numbers in $[0,1]$ is countable.

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Proof:

The set of all rational numbers in $[0, 1]$ is a subset of the set of rational numbers which is countable.

By our known result "Every subset of a countable set is countable."

Therefore, the set of all rational numbers in $[0, 1]$ is countable.

Next to prove an important result:

to says that "the set of all real numbers is uncountable."

By known result "Every superset of an uncountable set is uncountable".

It is enough to show that the set $[0, 1]$ is uncountable.

For this purpose, assume that every real number x can be expressed in decimal form as

$$x = a_0 . a_1 a_2 a_3 \ldots ,$$