Result:

1. The set of all positive rational numbers is countable.
2. The set of all negative rational numbers is countable.
3. The set Q of all rational numbers is countable.
4. The set $[0, 1]$ is uncountable.
5. The set of real numbers is uncountable.
6. The set of irrational numbers is uncountable.

1.6 Real Numbers

1.6A Theorem:

The set $[0, 1] = \{x / 0 \leq x \leq 1\}$ is uncountable.

Proof:

Suppose $[0, 1] = \{x / 0 \leq x \leq 1\}$ were ~~un~~countable.

Then $[0, 1] = \{x_1, x_2, \ldots\}$ were every number in $[0, 1]$ occurs among the x_i.

Expanding each x_i in decimals we have,

$$x = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \dots$$

where a_0 is an integer, and $a_1, a_2, \dots$ are all integers such that $0 \le a_i \le 9$, for all i.

This expression for a real number is unique except when the real number is a rational number of the form $\frac{p}{2^m 5^n}$, where p is an integer and m & n take any of the values $0, 1, 2, 3, \dots$.

In such a case, two decimal expansions are possible. (For example: Express $\frac{1}{2}$ either as 5000 or 0.4999...)

Conversely, every decimal of the form $a_0 . a_1 a_2 a_3 \dots$, where a_0 is an integer, $a_1, a_2, a_3, \dots$ are all integers such that $0 \le a_i \le 9$, for all i, is the decimal expansion of some real number.

Hence proved.

$x_1 = 0.a_1^1 a_2^1 a_3^1 \ldots$

$x_2 = 0.a_1^2 a_2^2 a_3^2 \ldots$

$\vdots$

$x_n = 0.a_1^n a_2^n a_3^n \ldots a_n^n \ldots$

$\vdots$

Let b_1 be any integer from 0 to 8 such that $b_1 \neq a_1^1$.

Then let b_2 be any integer from 0 to 8 such that $b_2 \neq a_2^2$.

In general, for each $n = 1, 2, \ldots$, let b_n be any integer from 0 to 8 such that $b_n \neq a_n^n$.

Let $y = 0.b_1 b_2 \ldots b_n \ldots$

Then, for any n, the decimal expansion for y differs from the decimal expansion for x_n since $b_n \neq a_n^n$.

Moreover, the decimal expansion for y is unique since no b_n is equal to 9.

Hence $y \neq x_n$ for every n and $0 \leq y \leq 1$, which is contraduction the assumption that every number in $[0, 1]$

Occurs among the x_i.
Hence $[0,1]$ is countable is wrong. This is contradiction.
Hence $[0,1]$ is uncountable.
Hence proof of the theorem.

1.6B Corollary:
The set R of all real numbers is uncountable.

Proof:
Given R be the set of all real numbers.
By our known theorem
"If B is an infinite subset of the countable set A, then B is countable".
If R were countable, then $[0,1]$ would be countable, this is contradicting to the fact that by our known theorem "The set $[0,1] = \{x / 0 \le x \le 1\}$ is uncountable".
Hence R is uncountable.

Definition: 1.6D
The cantor set K is the set of all numbers x in $[0,1]$ which have a ternary expansion without the digit 1.

1.7 Least upper bounds

1.7 A Defn:-

The Subset $A \subset R$ is said to be bounded above if there is a number $N \in R$ such that $x \leq N$ for every $x \in A$.

The Subset $A \subset R$ is said to be bounded below if there is a number $M \in R$ such that $M \leq x$ for every $x \in A$.

If A is both bounded below and bounded above, it is called A is bounded.

Ex: The set I of positive integers is bounded below but not bounded above. Hence I is not bounded.

Note: If A is bounded iff $A \subset [M, N]$ for some interval $[M, N]$ of finite length.

1.7B: Defn: If $A \subset R$ is bounded above, then N is called an upper bound for A if $x \leq N$ for all $x \in A$.

If $A \subset R$ is bounded below, then M is called a lower bound for A if

$M \leq x$ for every $x \in A$.

Note: To ~~ap~~ abbreviate upper bound and lower bound by u.b. and l.b. respectively.

1.7C Defn:

If $A \subset \mathbb{R}$ be bounded above. The number 'L' is called the least upper bound for A, if

(i) L is an upper bound for A,

(ii) no number smaller than L is an upper bound for A.

If $A \subset \mathbb{R}$ be bounded below. The number 'l' is called the greatest lower bound for A, if

(i) l is a lower bound for A,

(ii) no number greater than l is a lower bound for A.

Note:- To abbreviate "least upper bound" as l.u.b and "greatest lower bound" as g.l.b.

1.7 D Least upper Bound Axiom:

If A is any nonempty subset of R that is bounded above, then A has a least upper bound in R.

1.7 E Theorem:

If A is any nonempty subset of R that is bounded below, then A has a greatest lower bound in R.

Proof: Given $A \subseteq R$ is bounded below.

To prove that: A has a greatest lower bound in R.

Let $B \subset R$ be the set of all $x \in R$ such that $-x \in A$.

i.e, the elements of B are the negatives of the elements of A.

If M is a lower bound for A, then $-M$ is an upper bound for B.

For, if $x \in B$ then $-x \in A$ and so $M \leq -x$,

$\Rightarrow x \leq -M$.

Hence B is bounded above so that, By our known theorem "If A is any nonempty subset of R that is bounded above, then A has a least upper bound in R", we have, B has a l.u.b.

If Q is the l.u.b. for B then $-Q$ is the g.l.b for A.

Hence A has a greatest lower bound in R

Hence proved the theorem.

Unit – 2

4 – Limits and Metric Spaces:

4.1 Limit of a function on the real line

Defn: Limit:

Let 'C' and 'L' be real numbers. The function f has limit L as x approaches c if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta$$

$$\Rightarrow |f(x) - L| < \varepsilon$$