1. Elasticity

Elasticity

Elasticity is the property by virtue of which material bodies regain their original shape and size after the external deforming forces are removed. when an external force acts on a body, there is change in its length, shape and volume. The body is said to be strained.

when this external force is removed, the body regains its original shape and size. Such bodies are called elastic bodies. steel, glass ivory, quartz etc, are elastic bodies. The bodies which do not regain their original shape and size are called plastic bodies. No body is either completely elastic or completely plastic. The property of elasticity is different in different substances. Steel is more elastic than rubber. Liquids and gases are highly elastic.

Elasticity is defined as the property by which a body regains its original position when the forces are withdrawn.

The opposite of elasticity is plasticity. No substance is perfectly elastic or perfectly plastic.

Stress:

when a force is applied on a body, there will be relative displacement of the particles and due to the property of elasticity the particles tend to regain their original position. Stress is defined as the restoring force per unit area.

Normal stress:

Restoring forces per unit area perpendicular to the surface is called normal stress.

Tangential stress:

Restoring force parallel to the surface

per unit area is called tangential stress

Strain.

The ratio of the change in shape to the original shape is called strain.

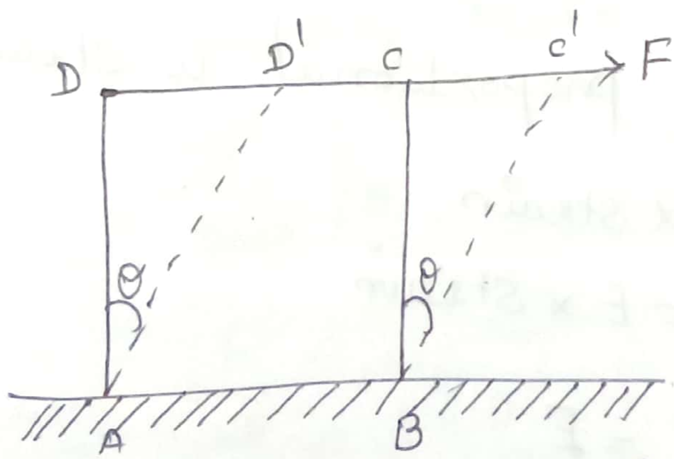
There are three types of strain:

Longitudinal strain.

The ratio of change in length to original length is called longitudinal strain (l/L)

Shearing strain:

Shearing strain is defined as the angle of shear measured in radians.



The surface AB is fixed and a force is applied parallel to the surface CD. So that the body is ~~sheared~~ shearing strain θ.

Volume strain.

The ratio of the change in volume to original volume is called volume strain $\left(\frac{v}{V}\right)$

Elastic limit.

The maximum stress up to which a body exhibits the property of elasticity is called elastic limit. If the applied force exceeds the maximum stress limit, the body does not regain its original position completely after the external forces are withdrawn.

Hooke's law.

It states that within the elastic limit stress directly proportional to strain.

$$\text{Stress} \propto \text{Strain}$$

$$\text{Stress} = E \times \text{Strain}$$

$$\frac{\text{Stress}}{\text{Strain}} = E$$

E is a constant called the modulus of elasticity.

(i) Young's modulus of elasticity (Y).

It is defined as the ratio of normal stress to longitudinal strain.

$$Y = \frac{\text{Normal stress}}{\text{Longitudinal strain}}$$

$$= \frac{F}{\text{area}} \Big/ \frac{l}{L} = \frac{FL}{al}$$

(ii) Modulus of rigidity (η).

It is defined as the ratio of tangential stress of shearing strain.

$$\eta = \frac{\text{Tangential Stress}}{\text{Shearing Strain}}$$

$$= \frac{F}{\text{area}} \Big/ \theta$$

$$= \frac{F}{a\theta}$$

(iii) Bulk modulus of elasticity (k).

It is defined as the ratio of normal stress to volume strain.

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$$K = \frac{\text{Normal stress}}{\text{Volume Strain}}$$

$$= F/\text{area} \Big/ \frac{v}{V}$$

$$= \frac{FV}{av} = \frac{PV}{v}$$

where p is the change in Pressure.

Compressibility = $1/K$

Elastic limit: is the maximum stress within which the body exhibits the property of elasticity. Below the elastic limit, the body regains its original position or shape when the deforming force is removed.

Beyond the elastic limit, the body does not regain completely its original position even though the external force is withdrawn.

Elastic fatigue.

If a body is continuously subjected to stress and strain, it gets fatigued. Consider two torsional pendulums A and B having similar wires. A is initially set into vibration. After A has come to rest, both the pendulums A and B are set into vibration simultaneously. It will be found that due to elastic fatigue, A comes to rest earlier than B.

Poisson's Ratio.

whenever a body is subjected to a force in a particular direction, there is change in dimensions of the body in the other two perpendicular directions. This is called lateral strain. Lateral strain is proportional to the size of the body in the other two perpendicular directions.

Let α be the Longitudinal strain per unit stress and β the lateral strain per unit stress per unit stress, within the elastic limit

$$\beta \propto \alpha$$

or $$\beta = \sigma \alpha$$

$\therefore$ poission's ratio

$$\sigma = \frac{\beta}{\alpha}$$

Therefore, poisson's ratio is defined as the ratio of lateral strain per unit stress to the longitudinal strain per unit stress.

Workdone in Deforming a body.

Whenever a body is deformed by the application of external forces, the body gets strained. The work done is stored in the body in the form of energy and is called the energy of strain. There are three types of strain

(i) Longitudinal strain

(ii) shearing "

(iii) Volume "

(i) Longitudinal Strain.

Consider a wire of length L, area of cross-section a and Young's modulus of elasticity Y. Let l be the increase in length when a stretching force F is applied.

Therefore, workdone

$$\int dW = \int_0^l F \cdot dl$$

But $$Y = \frac{F/a}{l/L}$$

or $$F = \frac{Yal}{L}$$

$$W = \int_0^l \left(\frac{Ya}{L}\right) l \, dl = \frac{Yal^2}{2L}$$

$$W = \frac{1}{2}\left(\frac{Yal}{L}\right) l = \frac{1}{2}(F)l$$

Workdone per unit Volume,

$$w = \frac{W}{V} = \frac{W}{aL}$$

$$w = \frac{Fl}{2aL} = \frac{1}{2}\left(\frac{F}{a}\right)\left(\frac{l}{L}\right)$$

$$w = \frac{1}{2} \times \text{Stress} \times \text{Strain}$$

(ii) Shearing Strain.

Consider a cube of side L. when a tangential force F is applied to the upper face of the cube (Keeping the lower face fixed), the cube is Sheared through an ~~ag~~ angle ϕ. If the tangential stress is T,

$$\eta = \frac{T}{\phi}$$

$$T = \eta \phi$$

Total tangential force

$$= Ta = \eta a \phi$$

Workdone $= \int F \cdot dl$

Here $dl = L\, d\phi$

Workdone $= \int_0^{\phi} T (L\, d\phi) = \int_0^{\phi} \eta a L \phi \, d\phi$

$$W = \frac{\eta a L \phi^2}{2}$$

Workdone per unit Volume

$$w = \frac{\eta a L \phi^2}{2aL} = \frac{\eta \phi^2}{2} = \frac{(\eta \phi)\phi}{2}$$

$$w = \tfrac{1}{2} (T) \phi = \tfrac{1}{2} \text{ Stress} \times \text{Strain}.$$

(iii) Volume strain:

Consider a cube of volume V, area of cross-section A and length L. When a normal stress p is applied the change in volume = v.

$$\text{Workdone} = \int_0^v p \cdot dv$$

$$k = \frac{pV}{v}$$

$$p = \frac{kv}{V}$$

$$W = \int_0^v \frac{kv}{V} \, dv$$

$$W = \frac{kv^2}{2V} = \frac{1}{2}\left(\frac{kv}{V}\right)v$$

$$W = \frac{1}{2} \, p \times v$$

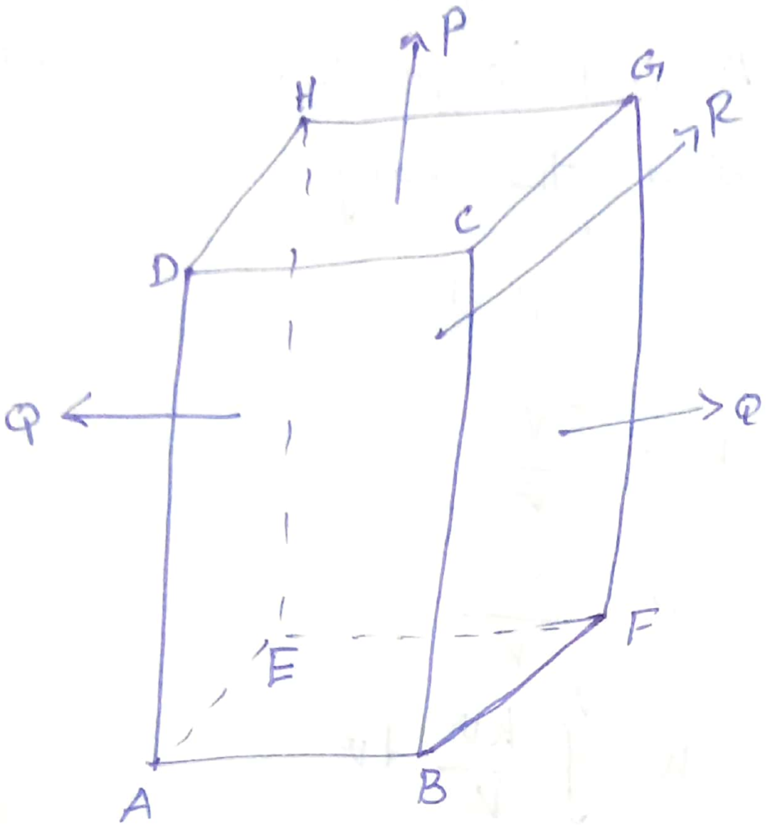
Workdone per unit volume,

$$w = \frac{p \times v}{2V}$$

$$= \frac{1}{2}(p)\left(\frac{v}{V}\right)$$

$$w = \frac{1}{2} \times \text{Stress} \times \text{Strain}$$

Bulk Modulus (Relation between K, Y and σ)



Consider a block of rectangular cross Section having length l, breadth b and thickness t. Let the block be subjected to outward stress as shown in Fig. Here P is the stress acting on the faces ABFE and CDHG. Here Q is the stress acting on the faces AEHD and BCGF and R is the stress acting on the faces ABCD and EFGH.

Each stress produces an extension in its own direction and a lateral contraction

in the other two perpendicular directions.
Let α be the longitudinal strain per unit stress and β ~~and~~ the lateral strain per unit stress. poisson's ratio $\sigma = \frac{\beta}{\alpha}$

Increase in length due to the stress p is $p\alpha l$ and corresponding contraction in breadth and thickness will be $p\beta b$ and $p\beta t$ respectively.

Similarly, due to the stress Q, increase in breadth is $Q\alpha b$ and corresponding decrease in length and thickness will be $Q\beta l$ and $Q\beta t$ respectively. Also due to the stress R, increase in thickness is $R\alpha t$ and decrease in length and breadth will be $R\beta l$ and $R\beta b$

Final length $= l + p\alpha l - Q\beta l - R\beta l$

Final breadth $= b + Q\alpha b - p\beta b - R\beta b$

Final thickness $= t + R\alpha t - p\beta t - Q\beta t$

Final volume $= lbt\,[1 + p\alpha - (Q+R)\beta] \times [1 + Q\alpha - (P+R)\beta] \times [1 + R\alpha - [P+Q]\beta]$

$= lbt[1+(P+Q+R)\alpha - 2\beta(P+Q+R)]$

$= lbt[1+(P+Q+R)(\alpha-2\beta)]$

Change in volume $= lbt(P+Q+R)(\alpha-2\beta)$

If the stresses are equal $P=Q=R$

chang in volume $= lbt(3P)(\alpha-2\beta)$

$$\text{Strain} = \frac{lbt(3P)(\alpha-2\beta)}{lbt}$$

$$= 3P(\alpha-2\beta)$$

$$\text{Bulk modulus } K = \frac{P}{3P(\alpha-2\beta)} \quad \ldots (i)$$

or $$K = \frac{1}{3(\alpha-2\beta)}$$

$$K = \frac{1}{3\alpha\left(1-\frac{2\beta}{\alpha}\right)}$$

$$K = \frac{1}{3\alpha(1-2\sigma)} \quad \ldots (ii)$$

But $$\frac{1}{\alpha} = Y$$

$$K = \frac{Y}{3(1-2\sigma)} \quad \ldots (iii)$$

— x —

Relation between Elastic Constants (Y, η, k and σ)

The various equations are

$$Y = \frac{1}{\alpha} \quad \ldots (1)$$

$$k = \frac{1}{3(\alpha - 2\beta)} \quad \ldots (2)$$

$$\eta = \frac{1}{2(\alpha + \beta)} \quad \ldots (3)$$

From equ (2) and (3)

$$\alpha - 2\beta = \frac{1}{3k} \quad \ldots (4)$$

$$2\alpha + 2\beta = \frac{1}{\eta} \quad \ldots (5)$$

Adding (4) and (5) Simplifying

$$3\alpha = \frac{1}{3k} + \frac{1}{\eta}$$

$$\frac{3}{Y} = \frac{1}{3k} + \frac{1}{\eta} \quad \ldots (6)$$

From equ (6)

$$Y = \frac{9\eta k}{\eta + 3k} \quad \ldots (7)$$

$$\eta = \frac{3kY}{9k - Y} \quad \ldots (8)$$

$$k = \frac{Y\eta}{9\eta - 3Y} \quad \ldots (9)$$

From equation (7) (8) and (9) values of Y, η and k can be calculated if any two value are known

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From equation (3)

$$\eta = \frac{1}{2(\alpha+\beta)} = \frac{1}{2\alpha(1+\beta/\alpha)}$$

$$\eta = \frac{Y}{2(1+\sigma)} \quad \ldots (10)$$

or $$\sigma = \left(\frac{Y}{2\eta}\right) - 1 \quad \ldots (11)$$

Multiplying equation (4) by 2 Substracting it from equation (V) we get

$$6\beta = \frac{1}{\eta} - \frac{2}{3k}$$

$$\beta = \frac{3k - 2\eta}{18\eta k} \quad \ldots (12)$$

The relation for poisson's ratio can be found in terms of k and η.

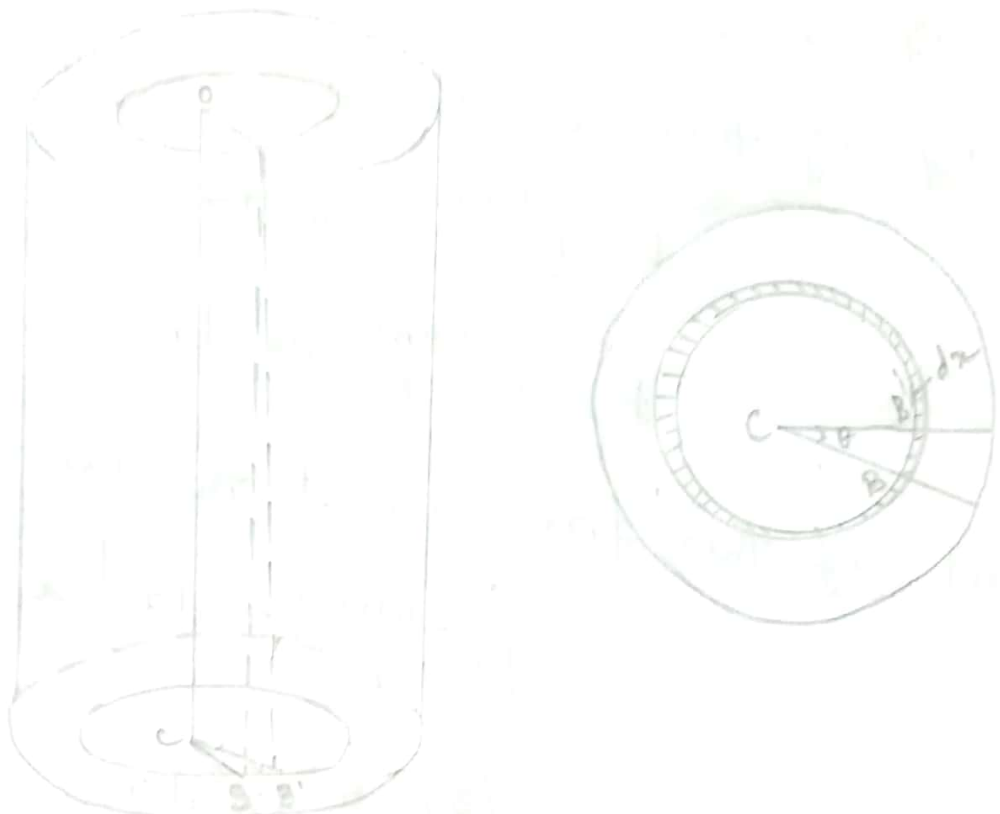
$$\sigma = \frac{\beta}{\alpha} = \beta Y = \left(\frac{3k-2\eta}{18\eta k}\right)\left(\frac{9\eta k}{\eta+3k}\right)$$

$$\sigma = \frac{3k-2\eta}{2(\eta+3k)} \quad \ldots (13).$$

Twisting of a cylinder

Consider a cylindrical rod of length l and radius r. Its upper end is clamped and the rod is twisted by applying a couple to its lower end in a plane perpendicular to its length. The rod is said to be under torsion. A reaction is set up due to its property of elasticity and a restoring couple equal and opposite to the twisting couple is produced.

All the particles in the lower end are shifted through the same angle θ but the linear displacement of a particle near the rim is more than the particle near the centre. Hence Shearing angle ϕ is more for the particles near the rim than near the axis of the cylinder.



Consider a cylinderical shell of radius x and radial thickness dx [Fig]

$$BB' = x\theta = l\phi$$

(or) $$\phi = \left(\frac{x\theta}{l}\right) \quad \ldots \quad (1)$$

ϕ is maximum at the rim and zero at the axis

Shearing Stress $= T$

$$\eta = \frac{T}{\phi}$$

$$\therefore \quad T = \eta\phi = \frac{\eta x \theta}{l}.$$

Area of cross section of the shell $= 2\pi x\,dx$

Shearing of force on the shell $= T \times$ area

$$F = \left(\frac{\eta x \theta}{l}\right) 2\pi x\,dx$$

$$F = \left(\frac{2\pi\eta\theta}{l}\right) x^2 dx$$

Moment of this force about the axis OC,

$$= \left[\left(\frac{2\pi\eta\theta}{l}\right) x^2 dx\right] x$$

$$= \left(\frac{2\pi\eta\theta}{l}\right) x^3 dx$$

Total twisting couple for the whole cylinder

$$\tau = \int_0^r \left(\frac{2\pi\eta\theta}{l}\right) x^3 dx = \left(\frac{\pi\eta r^4}{2l}\right)\theta \quad \text{---- (2)}$$

Let the couple per unit twist be C

$$\tau = C\theta$$

Comparing ~~eq~~ (1) and (2)

$$C = \left(\frac{\pi\eta r^4}{2l}\right) \quad \text{---- (3)}$$

Couple per unit twist C is also called the torsional rigidity of the material of the wire

Hollow Cylinder.

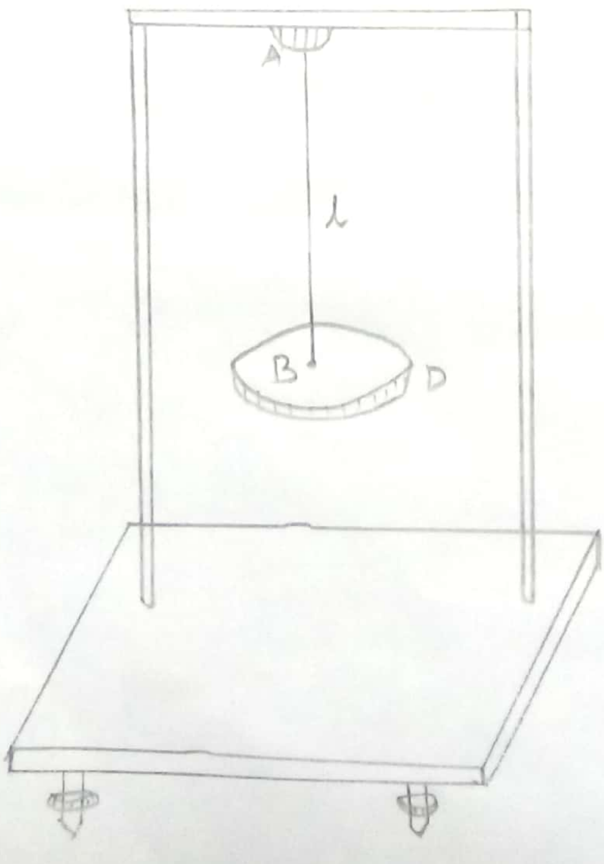
Consider a hollow cylinder of inner radius r_2 and outer radius r_1

$$\tau' = \int_{r_2}^{r_1}\left(\frac{2\pi\eta\theta}{l}\right)x^2dx = \left(\frac{\pi\eta\theta}{2l}\right)(r_1^4 - r_2^4)$$

$$c' = \frac{\pi\eta}{2l}(r_1^4 - r_2^4) \quad \ldots\ldots (4)$$

Torsion pendulum.

A torsion pendulum consists of a rigid metallic frame. D is a solid circular disc of moment of inertia I, mass M and radius R. The wire AB of length l and the radius r is fixed at the end A and the lower and B is clamped to the centre of the disc D [Fig]



When the disc D is rotated about the axis AB, the wire AB gets twisted. The disc executes SHM. At any instant, the deflecting couple is equal to the restoring couple

$$\therefore I\alpha = -C\theta$$

$$I\frac{d^2\theta}{dt^2} = -C\theta$$

(the negative sign shows that the restoring couple is in the opposite direction to the deflecting couple)

$$\therefore \frac{d^2\theta}{dt^2} + \left(\frac{C}{I}\right)\theta = 0 \quad \ldots \ldots (1)$$

This is the equation of SHM

$$\omega^2 = \frac{C}{I}$$

$$\omega = \sqrt{\frac{C}{I}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{C}}$$

But $C = \frac{\pi\eta r^4}{2l}$

$$\therefore T = \sqrt{\frac{2Il}{\pi\eta r^4}} \quad \ldots \ldots (2)$$

$$\eta = \frac{8\pi I l}{T^2 r^4}$$

But $$I = \frac{MR^2}{2}$$

$$\eta = \frac{4\pi M R^2 l}{T^2 r^4} \quad \text{--- (3)}$$

From equation (iii), η can be calculated.

The value of the radius of the wire AB should be measured accurately because in the equation it occurs in the fourth power of the radius of the experimental wire.