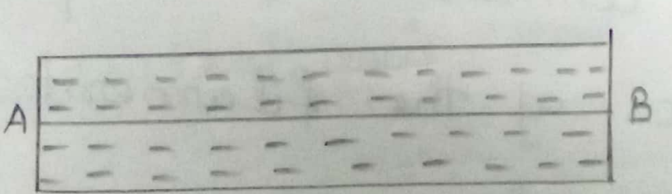
UNIT :2

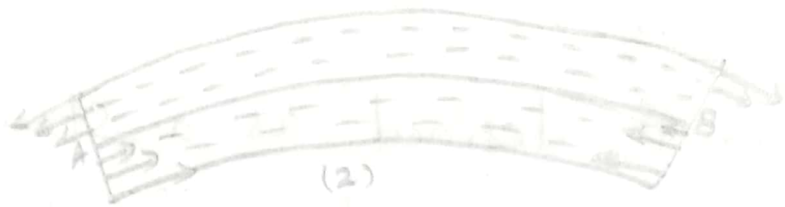
Bending of Beams.

A beam is defined as a structure of uniform cross section, whose length is large in comparison to its breadth and thickness. For such a structure, the shearing stress for any given cross section is negligible.

Beams are used in the construction of bridges or for purposes of supporting heavy loads. They are commonly used in the structure of multistoried buildings. A beam can be used in a horizontal position or as columns and pillars in a vertical position.

Consider a beam of uniform rectangular cross section. Let the beam be subjected to deforming forces so that it bends (Fig)

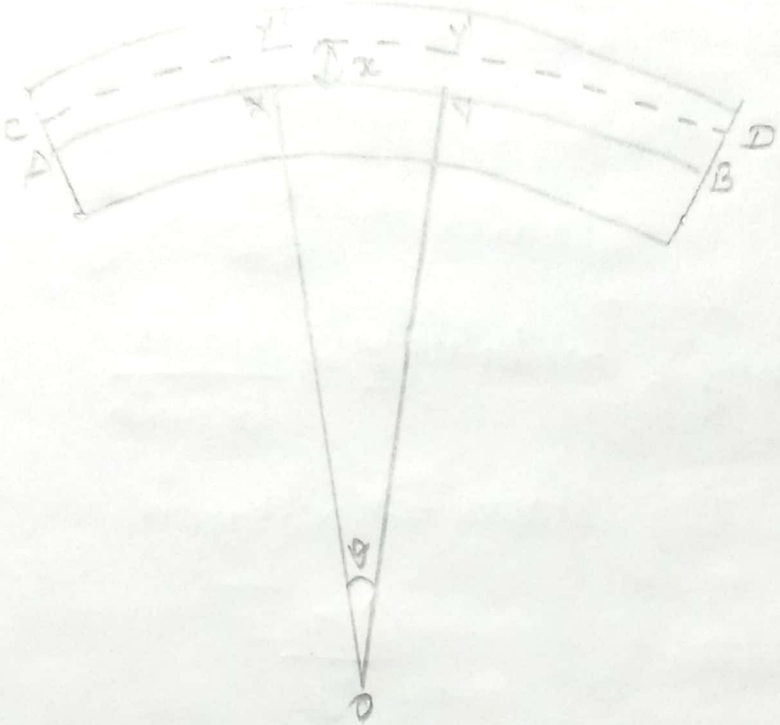




(2)

In the initial position of the beam, the various filaments constituting the beam are in parallel layers of equal length. when the deforming forces are applied, the beam bends as shown in fig(ii). Above the layer AB, the filaments are elongated while below AB they are compressed. The length of the layer AB remains unaltered. It is called the neutral axis. The surface containing the neutral axis and perpendicular to the plane of bending is called the neutral surface. Further the change in length of any filament (extension or contraction) is proportional to the distance of the filament from the neutral axis.

Bending Moment



Consider a beam under the action of deforming forces. Due to the property of elasticity of the material, a restoring couple acts on the beam. In the equilibrium position the bending couple is equal and opposite to the restoring couple. The moment of the restoring couple is called the bending moment.

Consider a filament $x'y'$ at a distance x from the neutral axis. Here $x'y'$ has been extended.

$$xy = R\theta$$

and $x'y' = (R+x)\theta$

$\therefore$ Increase in length $= x'y' - xy$

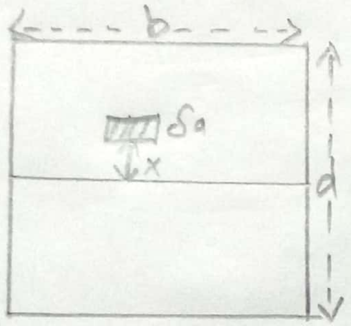
$$= (R+x)\theta - R\theta = x\theta$$

$$\text{Strain} = \frac{x\theta}{R\theta} = \frac{x}{R}$$

$$\text{Young's modulus } Y = \frac{\text{Stress}}{\text{Strain}}$$

$$\text{Stress} = Y \times \text{Strain}$$

$$= \frac{Yx}{R}$$



The cross section of the beam is shown in fig.

The force acting on the element of area of cross section δa

$$= \text{Stress} \times \delta a$$

$$F = \frac{Yx}{R} \times \delta a$$

The forces producing elongation act on the upper half outward and those producing

Contraction act on the lower half inward. These two forces constitute a Couple.

Moment of the force about the neutral axis

$$= \left[\frac{Yx}{R} \times \delta a\right] \cdot x = \left(\frac{Yx^2}{R}\right)\delta a$$

The moment of all the forces about the neutral axis.

$$= \Sigma \frac{Yx^2}{R} \delta a$$

$$= \frac{Y}{R} \Sigma x^2 \delta a$$

Here $\Sigma x^2 \delta a = ak^2 = I_g$

Where a is the area of cross section of the beam and k is the radius of gyration and $ak^2 = I_g$. I_g is called geometrical moment of inertia of the beam. Hence Bending moment

$$= \frac{Y I_g}{R}$$

Special Cases

1. Rectangular Cross Section

If the breadth of the beam is b and thickness is d, then

$$a = b \times d$$

and $$k^2 = \frac{d^2}{12}$$

$$\therefore \; I_g = ak^2 = \frac{(bd)\,d^2}{12}$$

$$= \frac{bd^3}{12}$$

2. Circular Cross Section.

Here $$a = \pi r^2 \text{ and } k^2 = \frac{r^2}{4}$$

$$I_g = ak^2 = \frac{\pi r^2 \times r^2}{4}$$

$$= \frac{\pi r^4}{4}$$

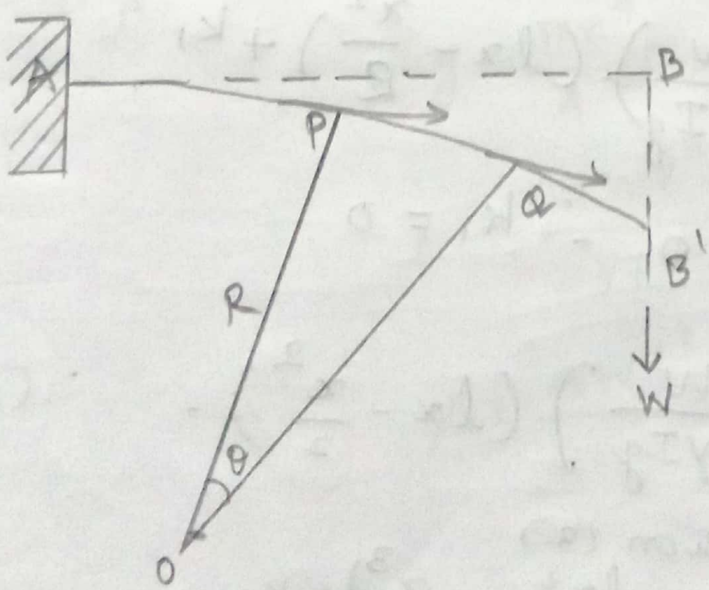
Flexural rigidity.

It is defined as the external bending moment required to produce a unit radius of curvature. Therefore, flexural Rigidity $= Y I_g$.

Cont

Cantilever

A Cantilever is a thin uniform bar fixed horizontally at one end and loaded at the other end. Let AB represent the neutral axis of the Cantilever of length l. The end A is fixed and the end B is loaded with a load W vertically downwards. The end B is displaced to the position B. The neutral axis of the Cantilever shifts to the position on AB'. It is assumed that the weight of the Cantilever is negligible.



Consider an element PQ, at a distance x from the end A and of radius of curvature R. Bending moment

$$= \frac{YI_g}{R}$$

Deflecting Couple $= W(l-x)$

For equilibrium

$$\frac{YI_g}{R} = W(l-x) \quad \cdots (1)$$

But $\frac{1}{R} = \frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{W(l-x)}{YI_g} \quad \cdots (2)$$

Integrating, equation (2)

$$\frac{dy}{dx} = \left(\frac{W}{YI_g}\right)\left(lx - \frac{x^2}{2}\right) + K_1$$

when $x = 0$ $\frac{dy}{dx} = 0$ $\therefore K_1 = 0$

$$\frac{dy}{dx} = \left(\frac{W}{YI_g}\right)\left(lx - \frac{x^2}{2}\right) \quad \cdots (3)$$

Integrating equation (3)

$$y = \frac{W}{YI_g}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + K_2$$

when $x = 0$ $y = 0$ $\therefore K_2 = 0$

$$y = \left(\frac{W}{Y I_g}\right)\left(\frac{l x^2}{2} - \frac{x^3}{6}\right) \quad \ldots (4)$$

For the depression of the free end $x = l$

$$y = \left(\frac{W}{Y I_g}\right)\left(\frac{l^3}{2} - \frac{l^3}{6}\right)$$

$$y = \frac{W l^3}{3 Y I_g} \quad \ldots (5)$$

Special cases.

1. Rectangular Cross-Section

$$I_g = bd^3/12$$

$$y = \frac{4 W l^3}{Y b d^3} \quad \ldots (6)$$

2. Circular Cross-section

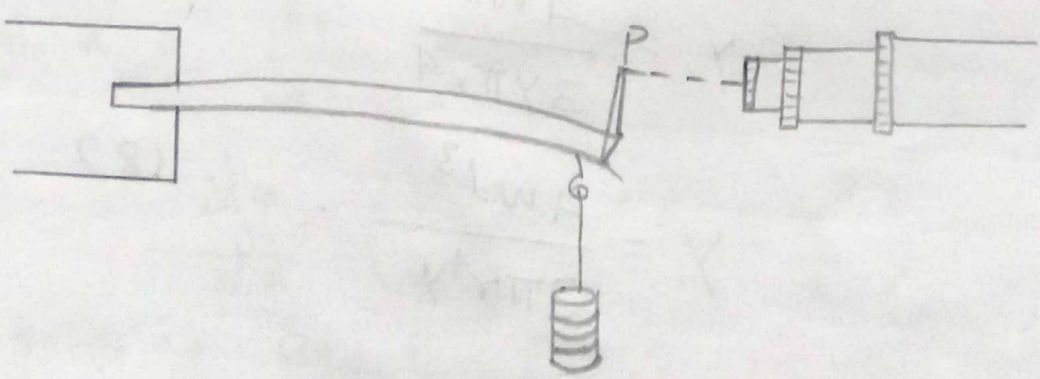
$$I_g = \frac{\pi r^4}{4}$$

$$y = \frac{4 W l^3}{3 Y \pi r^4} \quad \ldots (7)$$

$$y = \frac{4 W l^3}{3 \pi r^4 Y} \quad \ldots (8)$$

(1) Cantilever depression:

The given beam is clamped rigidly at one one. A weight hanger (H) is suspended at the free end of the beam. A pin (P) is fixed vertically by some wax at the free end of the beam. A travelling microscope (M) is focussed on the pin. The microscope is adjusted so that the horizontal cross-wire coincides with the tip of the pin the reading on the vertical scale of the microscope is noted. Then weights m, 2m, 3m, 4m etc..., are added to the weight-hanger. The microscope is adjusted each time to make the horizontal cross-wire coincide with the tip of pin



and the reading on the Vertical Scale of the microscope is noted in each case.

Observations are made for decreasing loads also. The results are tabulated as follows.

Load in kg	Microscope Reading			Depression for M (= 4m)	Mean depression for a load of M kg
	Load increasing	Load decreasing	Mean		
W					
W+m					
W+2m					
W+3m					
W+4m					
W+5m					
W+6m					
W+7m					

The mean depression (y) for a load M kg is found.

The length of the beam (l) between the Clamped end and the loaded end is measured. The mean breadth (b) of the beam and its mean thickness (d) are determined.

If y is the depression produced for a load of Mg, then.

$$y = \frac{Mgl^3}{3EAk^2} \quad \text{or} \quad E = \frac{Mgl^3}{3Ak^2y}$$

Now $Ak^2 = \frac{bd^3}{12}$ for a rectangular beam.

Hence $E = \frac{Mgl^3}{3(bd^3/12)Y} = \frac{4Mgl^3}{bd^3Y}$

The Young's modulus of the material of the beam is calculated using this relation.

Non-Uniform Bending:

The given beam is symmetrically supported on two knife-edges. A weight-hanger is suspended by means of a loop of thread from the point c exactly midway between the knife edges. A pin is fixed vertically at c by some wax. A travelling microscope is focussed on the tip of the pin such that the horizontal cross-wire coincides with the tip of the pin. The reading in the vertical traverse scale of microscope is noted. Weights are added in equal steps of m kg and the corresponding readings are noted. Similarly, readings are noted while unloading. The results are tabulated as follows.

The mean depression y is found for a load of M Kg. The length of the beam (l) between the knife-edges is measured. The breadth b and the thickness d of the beam are measured with a Vernier calipers and Screw gauge respectively.

Then $y = \frac{Wl^3}{48EAK^2}$ or $E = \frac{Wl^3}{48AK^2y}$

$$E = \frac{Mgl^3}{48 \times (bd^3/12) \times y} \quad (\because W = Mg \text{ and } AK^2 = bd^3/12)$$

$$E = \frac{Mgl^3}{4bd^3y}$$

(2) Uniform bending:

The given beam is supported symmetrically on two knife-edges A and B(fig). Two equal weight-hangers are suspended, so that their distances from the knife-edges are equal.

The elevations of the Centre of the beam may be measured accurately by using a single optic level (L). The front leg of the single optic lever rests on the Centre of the loaded beam and the hind legs are supported on stand. A vertical scale (S) and telescope (T) are arranged in front of the mirror and adjusted so that the reflected image of the scale in the mirror is seen through the telescope. The load on each hanger is increased in equal steps of m kg and the corresponding readings on the scale are noted. Similarly, readings are noted while unloading. The results are tabulated as follows.

Load in kg	Readings of the scale as seen in the telescope			Shift in reading for M kg
	Load increasing	Load decreasing	Mean	

The shift in scale reading for M kg is found from the table. Let it be S. If D = The distance between the scale and the mirror x = the distance between the front leg and the plane containning the two hind legs of the optic lever, then

$$y = Sx/2D$$

The length of the beam l between the knife-edges and a, the distance between the point of suspension of the load and the nearer knife-edge (AC = BD = a) are measured. The breadth b and the thickness d of the beam are also measured.

$$y = \frac{Wal^2}{8EAk^2} \text{ or } \frac{Sx}{2D} = \frac{Mgal^2}{8E(bd^3/12)}$$

Since $W = Mg$ and $Ak^2 = bd^3/12$

$$E = \frac{3Mgal^2 D}{Sxbd^3}$$

Pin and Microsope Method

The given beam is supported Symmetrically on two-~~ed~~ knife-edges A and B. Two equal Weight-hangers are suspended so that their distances from the knife-edges are equal. A pin is placed vertically at the centre of the beam. The tip of the pin is viewed by a microscope. The load on each hanger is increased in equal steps of m kg and the corresponding microscope readings are noted. Similarly, readings are noted while unloading. The results are tabulated as follows.

Load in kg	Readings of the microscope			Y For Mkg
	Load increasing	Load decreasing	Mean	

The mean elevation (y) of the centre for M kg is found. The length of the beam l between the knife-edges and a, the distance between the point of Suspension of the load and the nearer knife-edge ($AC = BD = a$) are measured. The breadth b and the thickness d of the beam are also measured.

$$y = \frac{Wal^2}{8EAK^2} = \frac{Mgal^2}{8E(bd^3/12)}$$

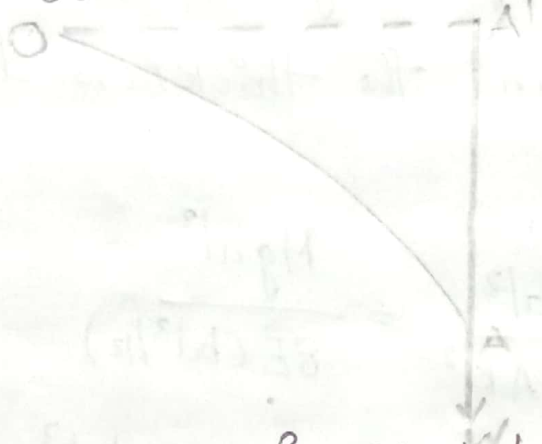
$$\left(\because W = Mg \text{ and } AK^2 = \frac{bd^3}{12}\right)$$

$$E = \frac{3Mgal^2}{2bd^3y}$$

Using the above formula, we can calculate the Young's modulus of the material of the beam.

Oscillations of a Cantilever.

Let OA be a cantilever of length l,



of negligible mass is slightly depressed and then released, the Cantilever will execute Simple harmonic motion about its original depressed position.

The depression of the loaded end of the Cantilever is

$$y = \frac{Wl^3}{3EAK^2}$$

or

$$W = \frac{3EAK^2}{l^3} y.$$

This must be equal to the elastic reaction of the Cantilever balancing it and hence directed opposite to it.

If M is the mass of the weight W and d^2y/dt^2, the acceleration (upwards), we have elastic reaction $= M \frac{d^2y}{dt^2}$

$$-M \frac{d^2y}{dt^2} = \frac{3EAk^2}{l^3} y$$

or
$$\frac{d^2y}{dt^2} = \frac{-3EAk^2}{Ml^3} y$$

But $\frac{3EAk^2}{Ml^3}$ = A constant.

The acceleration of mass M or the free end of the Cantilever is thus proportional to its displacement and is directed opposite to it.

It therefore, executes a S.H.M of time period T, given by

$$T = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

$$= 2\pi \sqrt{\frac{y}{\left(\frac{3EAk^2 y}{Ml^3}\right)}}$$

$$= 2\pi \sqrt{\frac{Ml^3}{3EAk^2}}$$

If the mass of the Cantilever is not negligible it can be shown that

$$T = 2\pi\sqrt{\frac{(M + \frac{1}{3}m)l^3}{3EAK^2}}$$

where m = mass of the Cantilever.

The mass of the cantilever can be eliminated by finding the periods T_1 and T_2 for two different masses M_1 and M_2 attached to cantilever at the same length. Then

$$T_1^2 = 4\pi^2 \frac{(M_1 + \frac{1}{3}m)l^3}{3EAK^2}$$

and $$T_2^2 = 4\pi^2 \frac{(M_2 + \frac{1}{3}m)l^3}{3EAK^2}$$

or $$T_2^2 - T_1^2 = \frac{4\pi^2 (M_2 - M_1) l^3}{3EAK^2}$$

$$E = \frac{4\pi^2 (M_2 - M_1) l^3}{3AK^2 (T_2^2 - T_1^2)}$$

Experiment: The given beam is rigidly clamped at O. A certain load of M_1 Kg is suspended from the other end A. The beam is set in transverse oscillations and the time for 25 oscillations is found. From this the period of oscillation T_1 is calculated. Similarly, the period T_2 with a load M_2 is found.

we have $$E = \frac{4\pi^2 (M_2 - M_1) l^3}{3Ak^2 (T_2^2 - T_1^2)}$$

For a rectangular bar, $Ak^2 = bd^3/12$

Hence
$$E = \frac{16\pi^2 l^3 (M_2 - M_1)}{bd^3 (T_2^2 - T_1^2)}$$

The length of the Cantilever l, the breadth b and depth d are measured. E is calculated using the above formula.

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